On the nonlinear dynamics of a position-dependent mass-driven Duffing-type oscillator: Lagrange and Newton equations' equivalence

Omar Mustafa*

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin 10 - Turkey, Tel.: +90 392 6301378; fax: +90 3692 365 1604.

Abstract: Using a generalized coordinate along with a proper invertible coordinate transformation, we show that the Euler-Lagrange equation used by Bagchi et al. [16] is in clear violation of the Hamilton's principle. We also show that Newton's equation of motion they have used is not in a form that satisfies the dynamics of position-dependent mass (PDM) settings.. The equivalence between Euler-Lagrange's and Newton's equations is now proved and documented through the proper invertible coordinate transformation and the introduction of a new PDM byproducted reaction-type force. The total mechanical energy for the PDM is shown to be conservative (i.e., $dE/dt = 0$, unlike Bagchi et al.'s [16] observation).

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The growing interests and/or research developments in the position-dependent mass (PDM) quantum mechanical systems described (mainly) by the von Roos Hamiltonian [1] (see also the sample of references [2–12] and related references cited therein) have inspired the relatively recent and rapid research attention in the PDM for classical mechanical systems (cf, e.g., [12–14, 16, 16, 17]). In fact, the interest in the PDM classical particles dates back to 1974 through the Mathews and Lakshmanan [18] study of the equation of motion

$$\left(1+\xi x^2\right)\ddot{x}-\xi x\,\dot{x}^2+\omega_\circ^2 x=0\ ;\ \xi\in\mathbb{R}, \tag{1}$$

where an overhead dote indicates a time derivative. This equation corresponds to a nonlinear oscillator, exhibiting amplitude-dependent simple harmonic oscillations [19], with the Lagrangian

$$L=\frac{1}{2}\left(\frac{\dot{x}^2-\omega_\circ^2 x^2}{1+\xi x^2}\right) \tag{2}$$

and the canonical linear momentum

$$p=m\left(x\right)\,\dot{x}\,;\ \ m\left(x\right)=\frac{1}{1+\xi x^2}. \tag{3}$$

In their attempt to reproduce the Lagrange equation in (1), Bagchi et al. [16] have proposed that in the absence of any external force term the Newton's equation of motion with PDM gets modified to

$$m\left(x\right)\ddot{x}+m^{'}\left(x\right)\dot{x}^2=0, \tag{4}$$

where the prime indicates spatial derivative. Which when compared with (1) would suggest a PDM function in the form of

$$m\left(x\right)=\frac{1}{\sqrt{1+\xi x^2}}, \tag{5}$$

on ignoring, of course, the presence of the harmonic term $\omega_\circ^2 x$ to effect such a comparison. They have also recollected Cruz et al's work [14] and used

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)-\frac{\partial L}{\partial x}=\tilde{R}\,;\ \ \tilde{R}=-\frac{1}{2}m^{'}\left(x\right)\dot{x}^2, \tag{6}$$

where $\tilde{R}$ is a reaction thrust (as named by Cruz et al. [14]). This would make their Lagrange's and Newton's equations of motion equivalent and consistent. However, this is an improper approach for the PDM-settings in the classical mechanical framework.

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*Electronic address: omar.mustafa@emu.edu.tr

A priory, the Lagrange's equation, used by Cruz et al [14], in (6) is a clear violation of the textbook Hamilton's least action principle, one of the most profound results in physics. Yet, one would notice that the Newton's equation in (4) is based on the conservation of the linear momentum, $dp/dt = \dot{p} = 0$, which is an improper approach for PDM settings. Mazharimousavi and Mustafa [17] have very recently shown that the quasi-linear momentum (as maned therein)

$$\Pi\left(x,\dot{x}\right)=\sqrt{m\left(x\right)}\dot{x} \tag{7}$$

is the conserved quantity (i.e., $\Pi\left(x,\dot{x}\right)=\Pi_0\left(x_0,\dot{x}_0\right)$ and $\dot{\Pi}\left(x,\dot{x}\right)=0$, where x_0 and $\dot{x}_0$ are the initial position and initial velocity of the PDM, respectively) and not the linear momentum (i.e., $p\left(x,\dot{x}\right)\neq p_0\left(x_0,\dot{x}_0\right)$, and $\dot{p}\left(x,\dot{x}\right)\neq 0$). In this communication, we fix this issue through the following arguments.

Let us consider a classical particle with *"unit mass"* in the generalized coordinate q moving with velocity $\dot{q}$ in a force-free field, $V(q) = 0$. Then the corresponding Lagrangian $L(q,\dot{q})$ is given by

$$L\equiv L\left(q,\dot{q}\right)=\frac{1}{2}\dot{q}^2, \tag{8}$$

and satisfies the Euler-Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0. \tag{9}$$

Under such assumptions, Eq.(9) would that $\ddot{q} = 0$. Which is in exact accord with Newton's law of motion for a free classical particle moving in the generalized coordinate q with a conserved generalized momentum

$$p_q=\frac{dq}{dt}=0\Longrightarrow \dot{q}=\dot{q}_0,$$

where $\dot{q}_0$ is the generalized initial momentum. Now, let the coordinate transformation

$$q\equiv q\left(x\right)=\int^x\sqrt{f\left(u\right)}du\Longrightarrow q'\left(x\right)=\sqrt{f\left(x\right)}\Longrightarrow \dot{q}\left(x\right)=\dot{x}\sqrt{f\left(x\right)} \tag{10}$$

represent a mapping from the coordinate q onto the coordinate x. As long as our choice of the coordinates is invertible (i.e., $det\left(\partial x_i/\partial q_i\right)\neq 0$, in general, which is the case under consideration here) then one can easily show that the Euler-Lagrange equation in (9) reads

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}\right]\left(\frac{\partial x}{\partial q}\right)=0\Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0. \tag{11}$$

This in fact follows immediately from the Hamilton's least action principle. The Euler-Lagrange equations are known to be coordinate invariant for they assume the same form in all coordinate systems, provided that the coordinates invertibility is secured. As such, the Lagrangian $L(q,\dot{q})$ in (8) would transform into

$$L\left(x,\dot{x}\right)=\frac{1}{2}f\left(x\right)\dot{x}^2 \tag{12}$$

with the corresponding Euler-Lagrange equation

$$f\left(x\right)\ddot{x}+\frac{1}{2}f^{'}\left(x\right)\dot{x}^2=0. \tag{13}$$

One may now go backwards and start with the Langrangian $L\left(x,\dot{x}\right)=\frac{1}{2}m\left(x\right)\dot{x}^2$ of a PDM particle $m(x) = f(x)$ moving in a force-free field, $V(x) = 0$, and use (11) to obtain

$$m\left(x\right)\ddot{x}=-\frac{1}{2}m^{'}\left(x\right)\dot{x}^2\Longrightarrow \sqrt{m\left(x\right)}\dot{x}=\sqrt{m\left(x_0\right)}\dot{x}_0\Longrightarrow \Pi\left(x,\dot{x}\right)=\Pi_0\left(x_0,\dot{x}_0\right). \tag{14}$$

This result would not only support our thesis in [17] on the conservation of the quasi-linear momentum (i.e., $\dot{\Pi}\left(x,\dot{x}\right)=0$) but also suggests the amendment that has to made for Newton's equation of motion (4) for PDM used by Bagchi et al. [16] and by Cruz et al. [14] in the absence of any external force term. In what follows we shall see that the "any external force" terminology holds true only for constant mass settings.

In this regard, we may express the linear momentum in terms of the quasi-linear momentum

$$p = m(x)\dot{x} = \sqrt{m(x)}\Pi(x,\dot{x}), \tag{15}$$

and cast Newton's equation of motion as

$$F_{ext}(x,\dot{x}) = F_{ext}(x) + R_{PDM}(x,\dot{x}) = \frac{dp}{dt}, \tag{16}$$

where $F_{ext}(x) = -\partial V(x)/\partial x = 0$ is the set of potential energy driven external forces, and $R_{PDM}(x,\dot{x})$ represents any feasible PDM-byproducted reaction-type force (of course, if it exists at all). To find out such a PDM-byproducted reaction force, $R_{PDM}(x,\dot{x})$, one would naturally find

$$\frac{dp}{dt} = \frac{d}{dt}[m(x)\dot{x}] = m(x)\ddot{x} + m^{'}(x)\dot{x}^2, \tag{17}$$

and

$$R_{PDM}(x,\dot{x}) = \frac{dp}{dt} = \frac{d}{dt}\left[\sqrt{m(x)}\Pi(x,\dot{x})\right] = \frac{1}{2}\frac{m^{'}(x)}{\sqrt{m(x)}}\dot{x}\Pi(x,\dot{x}) = \frac{1}{2}m^{'}(x)\dot{x}^2. \tag{18}$$

When (17) and (18) are substituted in the Newton's equation of motion in (16), along with $F_{ext}(x) = 0$, one obtains

$$\frac{1}{2}m^{'}(x)\dot{x}^2 = m(x)\ddot{x} + m^{'}(x)\dot{x}^2 \Longrightarrow m(x)\ddot{x} + \frac{1}{2}m^{'}(x)\dot{x}^2 = 0 \tag{19}$$

which is in exact accord with the Euler-Lagrange result in (14). Obviously, the equivalence between the Euler-Lagrange's and Newton's equations is now documented through the above "good" invertible coordinate transformation (i.e., $\partial x/\partial q \neq 0 \neq \partial q/\partial x$) and the introduction of the new PDM-byproducted reaction-type force $R_{PDM}(x,\dot{x})$ into Newton′s law of motion. Hereby, we may safely conclude that the PDM setting is nothings but a manifestation of some "good" invertible coordinate transformation that leaves the corresponding Euler-Lagrange equation invariant.

Under such textbook documentation settings, one would write the PDM Lagrangian as

$$L = T - V = \frac{1}{2}m(x)\dot{x}^2 - V(x), \; F_{ext}(x) = -\frac{\partial V(x)}{\partial x} \neq 0, \tag{20}$$

with the corresponding Euler-Lagrange equation

$$m(x)\ddot{x} + \frac{1}{2}m^{'}(x)\dot{x}^2 + \frac{\partial V(x)}{\partial x} = 0. \tag{21}$$

The Mathews and Lakshmanan [18] equation of motion in (1) is obtained if a PDM particle with $m(x) = \left(1+\xi x^2\right)^{-1}$ is subjected to move in an oscillator potential $V(x) = \frac{1}{2}m(x)\omega_\circ^2 x^2$ or a hypothetical potential $V(x) = -\frac{1}{2}m(x)\omega_\circ^2/\xi$ [20] (if such potential exists at all). Both potentials yield

$$m(x)\ddot{x} - m(x)^2\xi x\dot{x}^2 + m(x)^2\omega_\circ^2 x = 0. \tag{22}$$

Therefore, in the presence of an external periodic force with additional damping term and a quartic potential, the result in (22) would immediately suggest the following amendment to Eq. (11) of Bagchi et al.'s [16] to read

$$m(x)\ddot{x} - m(x)^2\xi x\dot{x}^2 + m(x)^2\omega_\circ^2 x + \lambda x^3 + \alpha\dot{x} = f\cos\omega t. \tag{23}$$

Which for a constant unit mass (i.e., $\xi = 0$) recovers a forced, damped Duffing oscillator. Yet, with $\dot{x} = y$ Eq.(12) of Bagchi et al. [16] should (taking $\dot{z} = \omega$) read

$$\dot{y} = \frac{\xi x\dot{x}^2 - \omega_\circ^2 x}{1+\xi x^2} + \left(1+\xi x^2\right)\left[f\cos z - \lambda x^3 - \alpha y\right]. \tag{24}$$

Moreover, one can show that the time rate of change of the total mechanical energy $E = T + V$ is given by

$$\frac{dE}{dt} = \left[m(x)\ddot{x} + \frac{1}{2}m^{'}(x)\dot{x}^2 + \frac{\partial V(x)}{\partial x}\right]\dot{x} \Longrightarrow \frac{dE}{dt} = 0, \tag{25}$$

indicating that the total mechanical energy is conserved (unlike equation (10) of Bagchi et al. [16]) and is the constant of motion. Therefore, the PDM-buproducted reaction force $R_{PDM}(x,\dot{x})$ turns out to be a conservative force with explicit dependence on both position and velocity. Should we assume that $m(x)$ and $m^{'}(x)$ are both positive valued functions, then $R_{PDM}(x,\dot{x})$ would act in the direction of the time rate of change of the linear momentum (i.e., $R_{PDM}(x,\dot{x}) = dp/dt = m^{'}(x)\,\dot{x}^2/2$). The consequences of the very existence of such a force are discussed by Mazharimousavi and Mustafa [17] through the use of the Euler-Langrange's equation (19) and the related examples reported therein.

For the sake of fairness and/or completeness, if one assumes that a PDM particle $m(x) = 1/\sqrt{1+\xi x^2}$ is moving under the influence of the set of mass-independent forces (as apparently suggested by Bagchi et al. [16]), only then Eq.(12) of Bagchi et al. would need a multiplicity of order 1/2 in the first term of $\dot{y}$ (as a consequence of our proposed amendment of Newton's low in (19)). This should not affect their numerical study of the propagation dynamics and the sensitivity of the PDM-index ξ to enhance phase transition from a limit cycle mode to a chaotic regime and the initiation of the complicated nature of bifurcation, etc.

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